Hackenbush, numbers, and combinatorial game theory

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1 Introduction

Hackenbush is a two-player game played with a pencil and paper (or other writing utensils). It begins with some graph consisting of red, blue, and green edges and a ground line, like the drawing below:

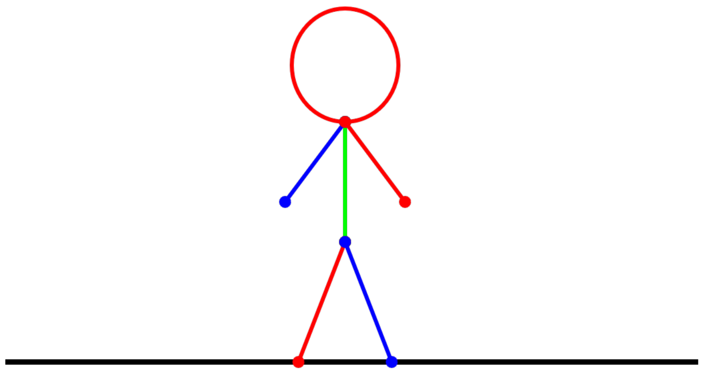


Figure 1: A Hackenbush board

The two players – which we will call **Left** and **Right** – take turns chopping off edges from the graph: Left can remove blue edges, Right can remove red edges, and either may remove green edges. Edges that are no longer connected the ground 'fall off' the board and disappear, and the first player who is unable to move loses.

The question we will aim to answer is: who wins?

First, is it possible to determine who will win? The answer to this question is *yes*, because Hackenbush, and many other games like chess, Go, and dots-and-boxes, is a *combinatorial game* – a two-player game with perfect information, no element of chance, and no draws by infinite repetitions of moves.

So who wins? Let's look at a few simple cases in Figure 2 below.

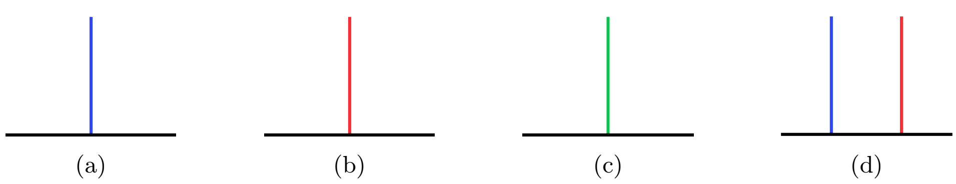


Figure 2: Games of class $\mathcal{L}$, $\mathcal{R}$, $\mathcal{N}$, and $\mathcal{P}$, respectively.

In (a), clearly Left wins, and similarly Right wins in (b). Whoever plays first in (c) will always chop down the green edge so that the second player has no moves, so the *first* player wins in (c). What about (d)? If

Left plays first, then Right makes the last move, and vice versa if Right plays first. So the *second* player always wins (d).

These make up the four outcome classes of combinatorial games: either Left can always force a win, Right can always force a win, the first player wins, or the second player wins. We call these classes *Positive*, *Negative*, *Fuzzy*, and *Zero*, and denote them by $\mathcal{L}, \mathcal{R}$, $\mathcal{N}$, and $\mathcal{P}$, respectively.

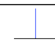
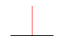
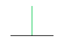
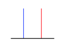
Class	Name	Example
$\mathcal{L}$	Left	
$\mathcal{R}$	Right	
$\mathcal{N}$	Fuzzy	
$\mathcal{P}$	Zero	

Table 1: Outcome classes with example games

The relation of the names of these classes to the signs of numbers is intentional, as it lends itself to a natural partial ordering of outcome classes by their favorability towards Left.

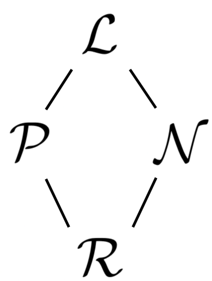


Figure 3: Hasse diagram of outcome classes

Left-winning games are greater Fuzzy and Zero games, which are both greater than Right-winning games, while Fuzzy and Zero games are incomparable – where the name Fuzzy comes from the fact that Fuzzy games are 'confused' with Zero.

What can we say about some slightly more complex cases, for example two blue edges stacked on top of each other?

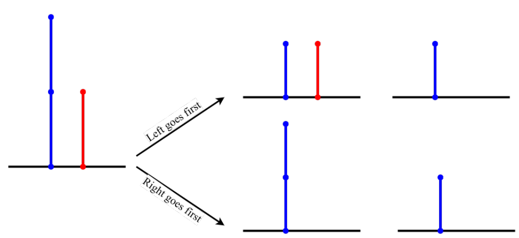


Figure 4: A slightly more complex game

Clearly this is better for Left than just a single blue edge. We know this because in the game with the stack of two blue edges and a single red edge, Left will always win if both players play optimally, whereas in the game with a single blue edge vs. a red edge, the second player wins. In fact, it takes exactly two of the red edges to 'cancel out' the advantage that Left has, which you can see by playing out the game.

We can quantify how much 'better' a position is for Left or Right by determining the *value* of a game. Before we do so, we must define what games are and what we can do with them.

2 Algebra of Games

We define a game recursively in terms of the moves that Left and Right may make.

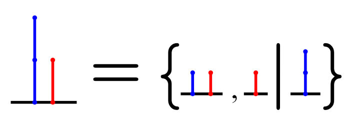
Definition 1. *Let G be a game. A* ***left option*** *of G is a game G^L such that Left may move directly from G to G^L (and likewise for a* ***right option*** *G^R).*

We denote the set of left options of G as $\mathcal{G}^L$, and the set of right options as $\mathcal{G}^R$, and we define the game G as its set of left options and right options:

$$G \coloneqq \{\mathcal{G}^L \mid \mathcal{G}^R\}$$

This definition is recursive, and the base case is the 0 game – the game where neither Left nor Right may move, which we write as $0 = \{\ \mid\ \}$. Since in the base case no options for either Left or Right, we can do induction on G assuming the induction hypothesis on any G^L or G^R without worrying about a basis, since it will come down to some statement about the members of the empty set, which will be vacuously true.

Using this notation for games, the game from Figure 4 can be written as follows (with the brackets for $\mathcal{G}^L$ and $\mathcal{G}^R$ omitted for brevity):



There is a natural way to decompose the game above into two disjoint games: the game with the two blue edges, and the game with one red edge. We say that it is the *disjunctive sum* of those two games.

Definition 2. *Let G and H be games. The* ***disjunctive sum*** *$G + H$ is defined as*

$$G + H = \{\mathcal{G}^L + H, \mathcal{G} + H^L \mid \mathcal{G}^R + H, \mathcal{G} + H^R\}$$

(where $\mathcal{G}^L + H$ is shorthand for $\{G^L + H : G^L \in \mathcal{G}^L\}$).

In other words, in $G + H$, each player either makes a move on the G or H, but not both. This operation is both commutative and associative – swapping the summands doesn't affect the outcome of the game, and likewise for grouping them together.

What does it mean for two games G and H to be equal? In some sense, if we could 'replace' G with H in any game and both players would be just as happy, then we can treat them as equal.

Definition 3. *Let G and H be games. $G = H$ if for all games X,*

$$o(G + X) = o(H + X),$$

where $o(G)$ denotes the outcome of G.

Note that this definition makes no reference to the type of game that G and H must be, which suggests that we can regard two games from completely different rulesets as equal. As an example, consider the game of Domineering: a board consists of a set of squares, and each player takes turns placing dominoes; Left place vertical dominoes while Right places horizontal dominoes. A Hackenbush game with just one red edge lends the same advantage to Right as a Domineering game with just two squares placed horizontally, so we

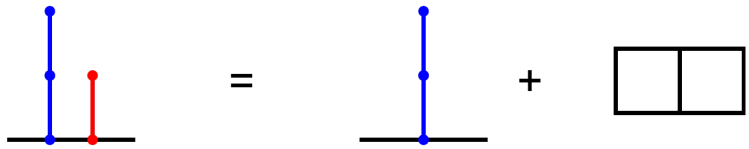


Figure 5: Domineering and Hackenbush

could replace the single red edge in the game above with the corresponding domineering game and have an equivalent game, as in Figure 5.

Now we will do some work to show that the set of games $\mathbb{G}$ along with $+$ is an abelian group.

Proposition 1. *Let G, H, and J be games, and suppose that $G = H$. Then $G + J = H + J$.*

Proof. Suppose that $G = H$. Then for any game X, $o(G + X) = o(H + X)$. Let $X' = J + X$. Then $o(G + (J + X)) = o(H + (J + X))$, so $o((G + J) + X) = o((H + J) + X)$ for all X; thus $G + J = H + J$. $\square$

We now define the additive identity of the set of games. Clearly, the zero game $0 = \{ \mid \}$ is an identity because it has no moves for either player, so adding it to any game will not affect the outcome. Are there others?

Proposition 2. *If $G \in \mathcal{P}$, then $o(G + X) = o(X)$ for all games X.*

Proof. We show that when Left plays first, $o(G + X) = o(X)$.

Suppose Left may win playing first in X. Then Left may win moving to some position in $G + \mathcal{X}^L$, and since Right loses moving first on both G and $\mathcal{X}^L$, we have that $o(G + X)$ is also a win for Left playing first.

Now suppose that Left loses playing first in X. Then for any move Left makes in either G or X, Right may respond in the respective component to force a win, so Left loses playing first in $G + X$.

Since the argument above may be repeated for Right playing first, we have that $o(G + X) = o(X)$ for all games X. $\square$

Since 0 is also in $\mathcal{P}$, it is natural to ask whether any game in $\mathcal{P}$ is equal to 0.

Proposition 3. *$G = 0$ if and only if $G \in \mathcal{P}$.*

Proof. Suppose that $G = 0$, then clearly G is a second player win.

For the converse, suppose that $G \in \mathcal{P}$, and let X be any game. By above we have that $o(G+X) = o(X)$, and since the zero game has no options for either player, $o(X) = o(0 + X)$; hence $o(G + X) = o(X) = o(0 + X)$ for all X. $\square$

This gives us not just one identity element, but a class of them – any element in $\mathcal{P}$ is an additive identity for the set of games. In particular, for Hackenbush, this includes all the games where there is symmetry in the game – the same moves are available for Left and Right – along with some other games like in Figure 6.

We can define the **negative** of a G as the game in which Left and Right are switched, which we write as $-G = \{-\mathcal{G}^R \mid -\mathcal{G}^L\}$. This gives us an additive inverse for G.

Proposition 4. *Let G be a game. Then $G - G \coloneqq G + (-G) = 0$.*

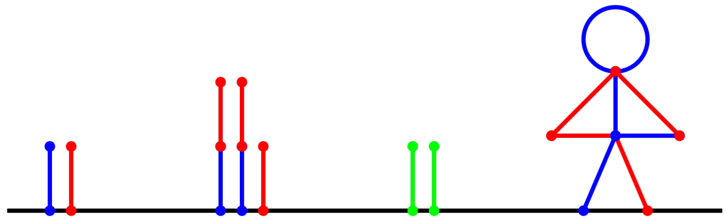


Figure 6: Some zero games

Proof. In $G + (-G)$, for any move that the first player moves in G, the second player may copy that move in $-G$ to win, and likewise if the first player moves in $-G$. Hence $G - G \in \mathcal{P}$, so $G - G = 0$. □

Proposition 5. *Let G and H be games. $G - H = 0$ if and only if $G = H$.*

Proof. Suppose that $G = H$. Then for all X, $o(G + X) = o(H + X)$. Hence $o(G - H) = o(H - H) = \mathcal{P}$, so $G - H = 0$. Now suppose that $G - H = 0$. Then the following holds:

$$\begin{aligned} G - H &= 0 \\ G + (-H + H) &= 0 + H \\ G &= H \end{aligned}$$

□

This gives us a way to see if two games G and H are equal: negate H, play it with G, and see if the second player will always wins.

From above, we have that the set of games along with + forms an abelian group: it has an identity element 0 (and any other game equal to it), an inverse element $-G$ for every game G, and a commutative and associative binary operation +. Furthermore, there is a partial ordering of games given by the partial ordering of their outcomes.

Definition 4. *$G \geq H$ if for all games X, $o(G+X) \geq o(H+X)$. Likewise, $G \leq H$ if for all X, $o(G+X) \leq o(H + X)$.*

This relation is reflexive, antisymmetric, and transitive, so it is a partial order, and it behaves just like we would expect $\geq$ to with ordinary numbers.

Now that we have some idea of how to compare games, we return to the topic of the value of a game.

3 Value of Games

We now begin to define the *value* of games. Intuitively, we can think of its value as the move advantage that Left has: the Hackenbush game with just one blue edge has a value of 1 since Left has one free move, the game with just one red edge has a value of -1, and any zero game has a value of 0.

(a) $1 = \{0 \mid \}$ (b) $-1 = \{ \mid 0\}$ (c) $0 = \{ \mid \}$ (d) $? = \{0 \mid 0\}$

Figure 7: Values of sample games

In terms of the notation we introduced previously, this gives us that $1 = \{0 \mid \}, -1 = \{ \mid 0\}$, and $0 = \{ \mid \}$, since in (a) the only move Left can make is to the zero game while Right has no moves, and similarly for (b) and (c).

Similarly, a blue 'stalk' with n edges has a value of $n = \{n-1 \mid \}$, and a red stalk with n edges a value of $n = \{ \mid -n+1\}$.

$2 = \{1 \mid \}$ $\quad$ $n = \{n-1 \mid \}$ $\quad$ $-2 = \{ \mid -1\}$ $\quad$ $-n = \{ \mid -n+1\}$

Figure 8: Single color stalks

Here we are somewhat abusing the notation for games introduced earlier – by $n-1$ do we mean the game that has value $n-1$ or the number $n-1$? – but we will just take it for granted that this is okay.

What about stalks with red and blue edges?

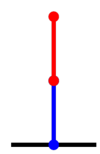


Figure 9: The game $\{0 \mid 1\}$

In Figure 9, Left always wins as she may always send the game to the zero game. By how much does she win? If we play two copies of the game against one red edge, we will find that the second player always wins. Does this mean $\{0 \mid 1\} = \frac{1}{2}$? Before we begin assigning numbers to games all willy-nilly, we first discuss what numbers are.

Numbers

For some historical context, Hackenbush in its Red-Blue form was originally created by John H. Conway as one of the ways in which the *surreal numbers* appear, which he simply refers to as numbers.

The construction of the surreal numbers begins as follows: Let L and R be sets of number such that no member of L is $\geq$ any member of R. Then $x = \{L \mid R\}$ is a number, where we denote $x^L \in L$ and $x^R \in R$.

Although this definition is recursive, we have a base case given where both L and R are the empty set (just like how we defined a game!). Since no member of $\varnothing$ is $\geq$ any member of $\varnothing$, this is indeed a number, which we call $0 = \{ \mid \}$.

Since new numbers are defined from old numbers, we have a way to think about when numbers are *born*. On day 0, only $0 = \{ \mid \}$ is born; the numbers $\{0 \mid\}$ and $\{ \mid 0\}$ are born by day 1 (but not $\{0 \mid 0\}$ as this breaks the rule that no $L \geq R$!). What names should we give these?

Conway defines that for numbers x and y, $x \geq y$ if and only if no $x^R \leq y$ and $x \leq$ no y^L, and $x \leq y$ if and only if $y \geq x$.

So $0 \leq 0$ by this definition, but how does it compare to $\{0 \mid\}$ and $\{ \mid 0\}$? Since $\{\mid\}$ has no options, we find that $\{ \mid 0\} < 0$ and $\{0 \mid \} < 0$, so it is natural to call these numbers -1 and 1, respectively. We may continue constructing numbers in this way and it turns out that we may construct all the real numbers in this manner and more! And they behave just we would expect numbers to with all the ordinary arithmetic operations.

Now we return to games. The rules for numbers above give us a way to associate certain games with numbers.

Definition 5. *Let* $G = \{\mathcal{G}^L \mid \mathcal{G}^R\}$ *such that for all* $G^L \in \mathcal{G}^L$ *and* $G^R \in \mathcal{G}^R$, $G^L < G^R$. *Then* G *is a number.*

Not all games are numbers. For example, consider the Hackenbush board with a single green edge. Whoever plays first removes the green edge, leaving the zero game. So this game is $\{0 \mid 0\}$, which is not a number as noted above.

Then which Hackenbush games are numbers?

Theorem 1. *Every Red-Blue Hackenbush game is a number.*

Proof. Let G be a Red-Blue Hackenbush game. We show that on chopping any blue edge of G, its value strictly decreases, and likewise on chopping a red edge of G, its value strictly increases.

Suppose Left moves from G to G^L by chopping off the blue edge e, and consider the difference game $G - G^L$.

If Right moves first, then for any move Right makes in $G - G^L$, Left may simply copy Right's move in the opposite component until Right makes a move in G for which there is no counterpart in $-G^L$, in which case Left can chop off e in G to reach a zero game, which Left wins. If Left starts, then Left moves to the zero game $G^L - G^L$ which Left also wins. Hence $G - G^L > 0$, so $G > G^L$.

The same argument may be used to show that Left wins $G^R - G$, and hence $G^R > G$. This implies that $G^L < G^R$ for any $G^L \in \mathcal{G}^L$ and $G^R \in \mathcal{G}^R$, so G is a number. □

So we see here that the numbers are a smaller set than the set of all the games. Only the games where Left and Right are in some sense *reluctant* to move because they will be in a worse position after they move are numbers. The Red-Blue-Green Hackenbush games are not numbers because removing green edges does not necessarily remove moves from Left or Right.

Since for any number x, all x^L are strictly less than the x^R, we would expect that x is strictly in between the two as well.

Theorem 2. *If* x *is a number, then* $x^L < x < x^R$ *for all* x^L *and* x^R

Proof. We show $x > x^L$ by playing the game $x - x^L$. If Left starts, then she may play to $x^L - x^L = 0$ and win. If Right starts, he may play to either $x^R - x^L$ or $x^R - x^{LL}$. In the first case since x is a number, $x^R > x^L$ so Left wins. In the latter case, by induction $x^{LL} < x^L$ and by definition of a number, $x^L < x^R$. Hence $x^R > x^{LL}$ and Left wins this game as well. A similar argument may be made to show $x^R > x$. □

Since new numbers are created from old numbers, this suggests a sort of strict order to the numbers as they are born.

On day zero we have $0 = \{ \mid \}$. On day one we have $\{ \mid 0\} < \{ \mid \} < \{0 \mid \}$ which we called $-1 < 0 < 1$, respectively.

On day two, since we have three numbers $(0, 1, -1)$ now available, there are a whole slew of possibile new numbers we could construct. But since $x^L < x < x^R$, if x^L and x^R have more than one number, then we really only have to worry about the *largest* such x^L and *smallest* x^R. So numbers like $\{-1, 0 \mid 1\}$ are really just $\{0 \mid 1\}$, which greatly reduces the numbers that we have to worry about.

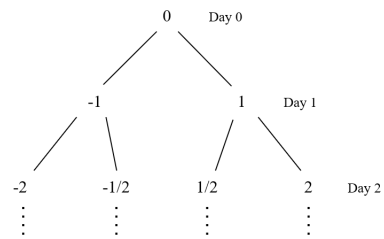


Figure 10: The numbers born on the first few days

What about numbers like $\{-1 \mid 1\}$? What should we call these?

Theorem 3 (Simplicity Rule)**.** *Let $G = \{\mathcal{G}^L \mid \mathcal{G}^R\}$ and suppose some number x satisfies $x \not\leq G^L$ and $x \not\geq G^R$ for all G^L and G^R, but no option of x satisfies those conditions. Then $G = x$.*

Proof. Consider the difference game $x - G$. If Left starts, then she may play to either some $x - G^R$ or some $x^L - G$. In the first case, since $x \not\geq G^R$ it follows that $x - G^R$ is negative so Right wins. In the second case, Right may play to some $x^L - G^L$. If $x^L - G^L > 0$, this implies that $\mathcal{G}^L < x^L < x < \mathcal{G}^R$ and x^L satisfies the conditions it should not have. Hence $x^L - G^L \leq 0$ and Right wins, so $x - G \leq 0$.

A similar argument can be made to show that $x - G \geq 0$, and hence $x - G = 0$, so $x = G$. □

Such an x is called the *simplest* number that 'fits into' G. What is meant by simplest? Recall that new numbers are defined as being in between some left set of numbers and right set of numbers 'born before' them. The theorem says that none of the options of x 'fit into' G, but x does. So x is the 'oldest' or 'first-born' number that fits into G.

In the case of $\{-1 \mid 1\}$ discussed before, this means that $0 = \{-1 \mid 1\}$. What about numbers like $\{0 \mid 1\}$? What names should we give these? Since we saw earlier that $\{0 \mid 1\} + \{0 \mid 1\} = 1$, we would like for this game to be $\frac{1}{2}$, and similarly it would be nice if we could continue to recursively and have $\{0 \mid \frac{1}{2^n}\} = \frac{1}{2^{n+1}}$. It turns out that these are good names for their respective numbers.

Definition 6. *Let $x^L < x^R$. The simplest number x between x^L and x^R is:*

- *The smallest magnitude integer that fits between x^L and x^R if some integer does.*
- *The number $\frac{i}{2^j}$ between x^L and x^R where j is minimal.*

These are the dyadic rationals, or rationals whose denominator is a power of two.

This gives us a way to recursively find the value of some arbitrary Red-Blue Hackenbush game: we can figure out the value of all the Left options and the value of all the Right options, and the value of the game itself will be simplest number between these.

Of course, this can become very tedious and although there are ways to reduce certain Hackenbush graphs like trees, in general it is NP-hard to compute the value of a Hackenbush game.

For Red-Blue stalks though, there is a nifty way to calculate its value due to Berlekamp [2]. If a stalk has n blue edges before it switches to a red edge for the first time, then treat the nth blue edge and first red edge as a 'binary point', then label every following blue edge as 1 and red edge as 0, and add an extra 1 at the end. Interpret everything after the point as a fraction in binary. Then the value of the stalk is $(n-1) + 0.$(the binary portion).

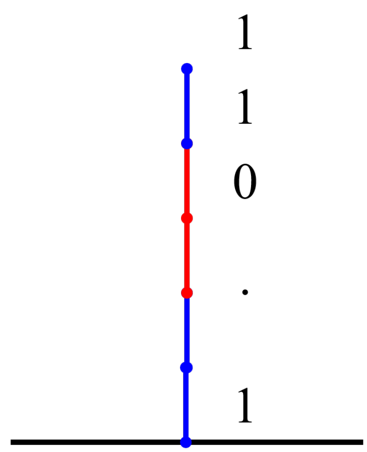


Figure 11: Berlekamp's algorithm for the game $\frac{11}{8} = \{|\}$

The rules are the same for stalks that start with red, the only difference being that the number is negative and we swap the rules for blue and red.

4 More Wacky Hackenbush Games

So far we have really only considered a small number of Hackenbush games – Red-Blue Hackenbush games where the Left and Right options are finite. In this section we will briefly explore the other games.

For example, what if we have infinite options, like $\{\frac{1}{4}, \frac{3}{16}, \frac{5}{32} \dots | \frac{1}{2}, \frac{3}{8}, \frac{5}{16} \dots\}$?

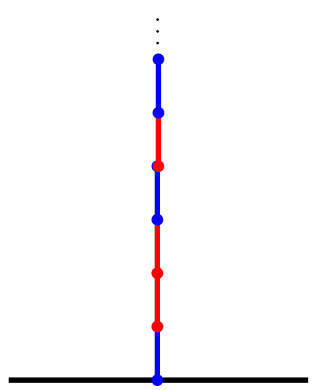


Figure 12: The infinite game $\{\frac{1}{4}, \frac{3}{16}, \frac{5}{32} \dots | \frac{1}{2}, \frac{3}{8}, \frac{5}{16} \dots\}$

First, is this even a game? The only condition we need to worry about here is the condition of 'no infinite draws by repetition', but since whenever Left or Right makes a move, the number of moves remaining is finite, we do not have to worry. It is also a number since all the left options are less than the right options. By Berlekamp's algorithm, we can see that this game has a value of $0.010101\dots$, which is $\frac{1}{3}$.

So the infinite Red-Blue Hackenbush games can give us the rationals that are not dyadic, and in a similar way we can also get all the real numbers from infinite Red-Blue Hackenbush stalks.

What about the infinite stalk with just Blue edges? This is also a number $\{1, 2, 3, \cdots |\}$ by the same reasoning as above – the number greater than all the natural numbers.

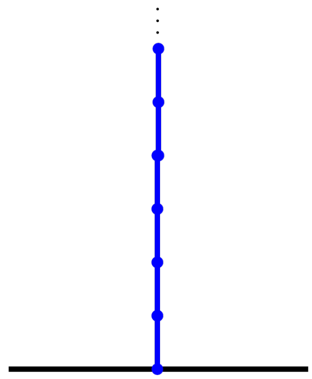


Figure 13: The infinite game $\{1, 2, 3, \cdots \mid \}$

Since this game has a countably infinite number of blue edges, we'll call this game ω, like Cantor's ω. We can put another blue edge on top of it to get $\omega + 1$, stack two of these on top of each other to get 2ω and all sorts of other infinite numbers.

Non-numbers

In the section about numbers, we sort of abandoned the Hackenbush games with green edges because these are not numbers. However, they are also very interesting in their own right.

We called the game with a single green edge $* = \{0 \mid 0\}$. What happens when we have a game with two green edges?

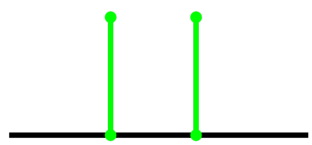


Figure 14: $* + * = 0$

Clearly whoever plays second will win, so $*+* = 0$. So $* = -*$ and $*$ is its own negative. This also illustrates an interesting point $- * + *$ is not a number, but 0 certainly is, so we can have games that are equal to numbers, but not numbers themselves!

So we have $0 = \{|\}$, $* = \{0 \mid 0\}$, and we may continue in this fashion and get games $*2 = \{0, * \mid 0, *\}$, $3* = \{0, *, *2 \mid 0, *, *2\}$, and $*n = \{0, *, ... * (n-1) \mid 0, *, ... * (n-1)\}$.

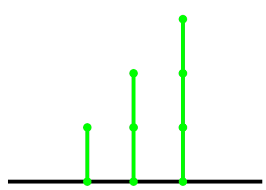


Figure 15: The nimbers $*$, $*2$, and $*3$

These are called *nimbers*, and just like $*+* = 0$, we have that $*n+*n = 0$ since the second player may simply copy whatever move the first player makes. Since any pair of equal sized nimbers is a 0 game, whichever player that reduces the game to only pairs of equal sized nimbers wins.

Take the $* + *2 + *3$ game above. Whichever move that the first player makes, the second player can always reduce the game to two equal stalks, so the second player wins and $*+*2+*3 = 0$. It turns out that the $+$ for nimbers behaves like bitwise $\oplus$ for their numbers (ignoring the $*$) in binary, so for example $01 \oplus 10 \oplus 11 = 00$ as expected.

What about games with a mix of green and red/blue edges, like in Figure 16? Left can always win this game

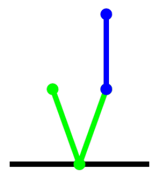


Figure 16: $\{0 \mid *\}$

so this game is definitely positive, but by how much? No matter how small of a positive rational number G we compare this game to, this game will be less than that. But this game is also not $*$ since playing it against $*$ results in a first player win, not a zero game. So this game is incomparable with $*$ and we call it $\uparrow$ and its counterpart with a red edge $\downarrow$. Although $\uparrow$ is incomparable with $*$, you can verify that $\uparrow + \uparrow > *$.

There are many more strange games with strange names, but we have no time to discuss them here so you can look to *Winning Ways for your Mathematical Plays* by Conway, Elwyn and Guy if you would like to explore more.

5 Miscellaneous

I wrote in the beginning that we would figure out who wins, but I have not really talked about how to actually play Hackenbush. So here are a few simple strategies, most of which come from *Winning Ways for your Mathematical Plays*:

- *Tweedledee-Tweedledum*: If you can get yourself into a position where you can simply copy the opponent's moves, then you've won since you will always have a move to make.
- *Enough Rope Principle*: If you are in a losing position, make the next position as complicated as possible, and if you are winning keep the position as simple as possible.
- *The green jungle slides off the purple mountain*: In an RBG Hackenbush game, the 'purple mountain' is the red and blue edges connected to the ground solely by other red and blue edges. Since the 'green jungle' will be infinitesimally small, it is always preferable to move on the 'green jungle' before you move on the 'purple mountain'.

By now you can probably see who wins in Figure 1, but in case you have not yet – the first player always wins, since removing the green edge results in a zero game. (Tweedledee-Tweedledum in action!)

6 My active component

For my active component, I made a little visualization of Red-Blue Hackenbush stalks on the number line. You can see which numbers are born on which days, though I limited it to days 0 - 10 because this already gives us so many numbers that the number line is a little hard to look at.

I wrote it in Python and used Bokeh to generate the visualization. To generate the numbers born by day n, I used the fact that after every day, the new numbers will be in between all the old numbers and then recursively computed them. Then I used Berlekamp's algorithm to turn these into Hackenbush stalks, and plotted them with Bokeh.

You can find it here and look at the source code here.

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